SOME ASPECTS OF STOCHASTIC CALCULUS IN RELIABILITY

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Abstract

In this paper we refer to some aspects regarding to the problem of the increase of the effectiveness of stand-by systems as a way in which the stochastic-approximation techniques can be applied in practice.

1 Introduction

It is a true that in the present-day technology, relaibility of equipment is increased by employing the method of *stand-by systems*, that is the introduction of extra components, units and entire assemblies. Thus, the porpose of the supplementary devices is to take over operation if the basic systems break down.

Dependind on the state of the stand-by equipment, can be distinguished loaded, nonloaded and partially loaded relief. In the case of loaded relief, the stand-by unit is in the same state as the operating unit and for this reason has the same intensity of breakdowns. In the partially loaded case, the stand-by device is loaded, but not so fully as the main equipment and for this reason has a different breakdown intensity. A stand-by unit that is not loaded does not, naturally, suffer breakdown. The spare wheel of an automobile is a typical example of nonloaded relief. Quite naturally, loaded and nonloaded relief are special cases of partially loaded relief.

In this sense, many and very important results are obtained by *Solovyev (1964)* and other are emphasized by *Gnedenko (1976)*. One of them is the problem of the increase of the effectiveness of stand-by systems which offer o specific application of the theory of stochastic processes.

2 Limit theorems in the study of the effectiveness of systems

Some problems concerning the increase of the effectiveness of stand-by systems, due especially to A. D. Solovyev and B. V. Gnedenko are discussed now.

There are enough situations when it is possible to have an entire device in reserve as, for example, a generator at a power station. Also it is possible to have in reserve a component of a system or even a single element. A question arises: *what is preferable, to have large units or single elements in reserve* ? An answer is given in the following theorem

Theorem 1. *If the switching of stand-by devices (units, elements, a.s.o.) is flawless, then both in the case of loaded and nonloaded relief, an increase in the scale of the stand-by system reduces non-breakdown operation of the whole system.*

To increase the effectiveness of stand-by systems, devices that have failed are repaired. Hence it is interesting to investigate the effect of repair on increasing the reliability. It is confined ourselves to the case of one basic and one reserve system.

The following conditions are supposed to be fulfilled:

i on breakdown of the basic device, the stand-by unit immediately takes up the load;

ii the device that has failed undergoes repair immediately;

iii the repairs fully restore the properties of the basic device that failed;
iv the repair time is a random variable with a distribution function $G(x)$;
v the repaired device becomes a stand-by unit;
vi the period of faultless operation of the device is random and is distributed in accord with the law $F(x) = 1 - e^{-\lambda x}$, $\lambda > 0$, for the basic device and in accord with the law $F_1(x) = 1 - e^{-\lambda_1 x}$, $\lambda_1 \geq 0$, for the stand-by device. In particular, if the stand-by unit is nonloa - ded then, $\lambda_1 = 0$ and if it is loaded then, $\lambda_1 = \lambda$.

Definition 1. *It is said that the system (basic unit plus stand-by unit) breaks down if both devices go out of commission at the same time.*

Let us denote by $P(x)$ the probability that the system will operate flawlessly for a time greater than x and also the Laplace transforms is introduced

$$g(s) = \int_0^\infty e^{-sx}\, dG(x), \quad \varphi(s) = -\int_0^\infty e^{-sx}\, dP(x)$$

Then the following result is obtained

Theorem 2. *Under the conditions $i - vi$ before, the probability $P(x)$ satisfies the following integral equation*

$$\begin{aligned} P(x) &= e^{-(\lambda+\lambda_1)x} + (\lambda+\lambda_1)\, e^{-\lambda x} \int_0^x e^{-\lambda_1 z}\, [1 - G(x-z)]\, dz + \\ &+ (\lambda+\lambda_1) \int_0^x \int_0^{x-y} e^{-(\lambda+\lambda_1)y - \lambda z}\, P(x-y-z)\, dG(z)\, dy. \end{aligned} \tag{1}$$

Proposition 1. *In terms of Laplace transforms, the solution of the equation (1) is given by the formula*

$$\varphi(s) = \frac{\lambda(\lambda+\lambda_1)[1 - g(\lambda+s)]}{(\lambda+s)[s + (\lambda+\lambda_1)(1 - g(\lambda+s))]} \tag{2}$$

Note 1. *By virtue of the properties of the exponential distribution, the result obtained can be immediately extended to the case when there are n operating devices and one stand-by unit. All devices have the same properties namely, they have the same distribution functions for operating time and repairs. It is necessary only to replace, in (1) and (2), λ by $n\lambda$.*

Now we can calculate the expectation of the time of flawless operation of the system and for this we consider

$$\left[\frac{d\varphi(s)}{ds}\right]_{s=0}.$$

We obtain successively

$$\begin{aligned} \left[\frac{d\varphi(s)}{ds}\right]_{s=0} &= \frac{[-\lambda(\lambda+\lambda_1)g(\lambda)][\lambda(\lambda+\lambda_1)(1-g(\lambda))]}{[\lambda(\lambda+\lambda_1)(1-g(\lambda))]^2} - \\ &- \frac{[\lambda(\lambda+\lambda_1)(1-g(\lambda))][(\lambda+\lambda_1)(1-g(\lambda)) + \lambda(1-(\lambda+\lambda_1)g(\lambda))]}{[\lambda(\lambda+\lambda_1)(1-g(\lambda))]^2} = \\ &= \frac{-\lambda(\lambda+\lambda_1)^2(1-g(\lambda))^2 - \lambda^2(\lambda+\lambda_1)(1-g(\lambda))}{[\lambda(\lambda+\lambda_1)(1-g(\lambda))]^2} \\ &= -\frac{[\lambda(\lambda+\lambda_1)(1-g(\lambda)][\lambda + (\lambda+\lambda_1)(1-g(\lambda))]}{[\lambda(\lambda+\lambda_1)(1-g(\lambda))]^2}. \end{aligned}$$

Therefore

$$m = -\left[\frac{d\varphi(s)}{ds}\right]_{s=0} = \frac{\lambda + (\lambda+\lambda_1)(1-g(\lambda))}{\lambda(\lambda+\lambda_1)(1-g(\lambda))}\,. \tag{3}$$

Now for a nonloaded stand-by system we have $\lambda_1 = 0$, so that it results

$$m_1 = \frac{\lambda + \lambda(1-g(\lambda))}{\lambda^2(1-g(\lambda))} = \frac{2-g(\lambda)}{\lambda(1-g(\lambda))} \tag{4}$$

while for a loaded stand-by system $\lambda_1 = \lambda$ and one gets

$$m_2 = \frac{\lambda + 2\lambda(1-g(\lambda))}{2\lambda^2(1-g(\lambda))} = \frac{3-2g(\lambda)}{2\lambda(1-g(\lambda))}\,. \tag{5}$$

But, in the most practical cases, the mean duration of repairs is considerably less than the mean time of flawless operation of the device. Thus the following limit theorems are necessary just to give a precise and rigorous meaning to the results obtained in these situations.

Let us suppose that the function $G(x)$ depends on a certain parameter ν and for any $\varepsilon > 0$,

$$1 - G_\nu(\varepsilon) \longrightarrow 0 \tag{6}$$

as $\nu \longrightarrow \infty$,

On the other hand, from (3), it is obtained that

$$g_\nu(\lambda) \longrightarrow 1 \tag{7}$$

as $\nu \longrightarrow \infty$.

The converse is also true because, if for any $s > 0$ we have the relation $g_\nu(s) \longrightarrow 1$, as $\nu \longrightarrow \infty$, then for any $x > 0$, we have

$$G_\nu(x) \longrightarrow 1$$

as $\nu \longrightarrow \infty$.

Let us denote

$$\alpha_\nu = \left(1 + \frac{\lambda_1}{\lambda}\right)(1 - g_\nu(\lambda))$$

or

$$\lambda + \lambda_1 = \frac{\lambda\,\alpha_\nu}{1-g(\lambda)}\,. \tag{8}$$

Now, we find

$$\varphi_\nu(\alpha_\nu s) = \frac{\lambda^2 \dfrac{1-g_\nu(\lambda+\alpha_\nu s)}{1-g_\nu(\lambda)}}{(\lambda+\alpha_\nu s)\left(s + \lambda\dfrac{1-g_\nu(\lambda+\alpha_\nu s)}{1-g_\nu(\lambda)}\right)}\,. \tag{9}$$

and the following theorem results

Theorem 3. *If the conditions (1), (2) and (3) to (9) hold then, by the condition (6), the flow of failures of a reduplicated system tends to the elementary case, given the choice of a proper unit of time.*

The effect of repair on the operational effectiveness of a system can be estimated. In this case it is natural to consider the ratio of the mean operational time of a system with repair to that without repair. From the formula (3) the former can be calculated and from the formula

$$a_0 = \frac{2\lambda + \lambda_1}{\lambda(\lambda + \lambda_1)}.$$

the latter.

The effectiveness of repair is now given by the equality

$$\begin{aligned} e_\nu &= \frac{\lambda + (\lambda + \lambda_1)(1 - g_\nu(\lambda))}{\lambda(\lambda + \lambda_1)(1 - g_\nu(\lambda))} \cdot \frac{\lambda(\lambda + \lambda_1)}{2\lambda + \lambda_1} = \\ &= \frac{\lambda + (\lambda + \lambda_1)(1 - g_\nu(\lambda))}{(2\lambda + \lambda_1)(1 - g_\nu(\lambda))} \end{aligned} \tag{10}$$

and

$$\frac{m_2(\nu)}{m_1(\nu)} \longrightarrow 0 \tag{11}$$

as $\nu \longrightarrow \infty$, so that the following theorem holds

Theorem 4. *Let us suppose that the conditions (1), (2), (3) to (10) and (11) are satisfied. Then for ν sufficiently large, the mean time of flawless operation of a system with stand-by relief is asymptotically equal to the mean time of the system under the assumption that*

$$G_\nu(x) = 1 - e^{-\nu x}.$$

References

Bharucha-Reid A.T. (1997). *Elements of the Theory of Markov Processes and Their Applications.* Dover Publications, Inc., Mineola, New York.

Gnedenko, B.V. (1976). *The Theory of Probability.* Mir Publisher, Moscow.

Orman, G.V. (1991). Some general aspects of the Markov dependence in the process of generation of the words (Preprint & Communic.). *The 20^{th} Conference on Stochastic Processes and their Application*, Nahariya, Israel.

Orman, G.V. (2000). *Măsură şi procese aleatoare.* Ed. Univ. "Transilvania", Braşov.

Orman, G.V. (2001). *Lectures on Stochastic Approximation Algorithms (Preprint).* Univ. "Gerhard Mercator", Duisburg, Germany.

Orman, G.V., Bârsan-Pipu, N., Vodă, V.G. (2002). On some properties of the homographic hazard rate variable. In H. Langseth, B. Lindqvist (Eds.). *Mathematical Methods in Reliability - MMR'2002*, pp.497-500. Norwegian University of Science and Technology. Tondheim, Norwey.

Orman, G.V. (2003). On Stochastic Approximation Methods in Reliability (Invited Lecture & Preprint). *ICRSA - 2003, University of South Carolina.* Columbia, USA.

Orman, G.V. (2003). *Handbook Of Limit Theorems And Stochastic Approximation.* "Transilvania" University Press, Brasov.

Solovyev, A. D. (1964). On Stand-by Systems Without Renewal, *Sbornik "Cybernetics at the Service of Comm."*, vol. 2, pp. 83-122.

Venter, J.H., Gastwirth, J.L. (1964). Adaptive statistical procedures in reliability and maintenance problems. *Technical Report No. 99, Department of Statistics*, Stanford University.

Wasan, M.T. (1967). Stochastic approximation and its applications. *American Mathematical Society Summer Seminar on "Mathematics of decision sciences".*

Wasan, M.T, (1969). *Stochastic Approximation.* Cambridge University Press.